

BOUNDARY JULIA THEORY FOR SLICE REGULAR FUNCTIONS

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ABSTRACT. The theory of slice regular functions is nowadays widely studied and has found its elegant applications to a functional calculus for quaternionic linear operators and Schur analysis. However, much less is known about their boundary behaviors. In this paper, we initiate the study of the boundary Julia theory for quaternions. More precisely, we establish the quaternionic versions of the Julia lemma, the Julia-Carathéodory theorem, the boundary Schwarz lemma, the Hopf lemma, and the Burns-Krantz rigidity theorem for slice regular self-mappings of the open unit ball $\mathbb{B} \subset \mathbb{H}$ and of the right half-space \mathbb{H}_+ . Especially, we find a new phenomenon that the classical Hopf lemma about $f'(\xi) > 1$ at the boundary point may fail in general in quaternions, and its quaternionic variant should involve the Lie bracket reflecting the non-commutative feature of quaternions.

1. INTRODUCTION

The celebrated Julia lemma [27] and the Julia-Carathéodory theorem [7, 8] for holomorphic self-mappings of the open unit disc $\mathbb{D} \subset \mathbb{C}$ and of the right half complex plane $\mathbb{C}^+ = \{z \in \mathbb{C} : \operatorname{Re}(z) > 0\}$ play an important role in the theory of hyperbolic geometry, complex dynamical systems, and composition operators; see, e.g., [35, 14, 34]. These two theorems together with the boundary Schwarz lemma [26] are powerful tools in the theory of iterating holomorphic self-mappings, their fixed points and boundary behaviors; see, e.g., [35, 14]. There are two canonical different approaches to the study of these two theorems. The usual one is the function-theoretic approach which has a strongly geometric character and depends ultimately on the Schwarz lemma and involves an asymptotic version of the Schwarz lemma, known as Julia's lemma. D. Sarason initiated the study of the Julia-Carathéodory theorem via a Hilbert space approach, which puts insight from a different perspective. In that treatment, Julia's lemma emerges as a consequence of the Cauchy-Schwarz inequality; see [32, 33] for more details.

There are many extensions for these results to higher dimensions for holomorphic functions. The Julia-Carathéodory theorem for holomorphic self-mappings on the open unit ball $\mathbb{B}_n \subset \mathbb{C}^n$ are studied by M. Hervé [25] and by W. Rudin [31]; see also [4] for the bidisk version and [1] for the polydisc version. Recently a variant of the Julia-Carathéodory theorem for infinitesimal generators has been investigated in [6]. However, there are no analogous results for other classes of functions, such as regular functions in the sense of Cauchy-Fueter and slice regular functions. A

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great challenge may arise from extensions to the setting of quaternions due to non-commutativity.

The purpose of the present article is to generalize the Julia lemma and the Julia-Carathéodory theorem as well as the boundary Schwarz lemma to the setting of quaternions for slice regular functions of one quaternionic variable.

The theory of slice regular functions is initiated recently by Gentili and Struppa [21, 22]. It is significantly different from that of regular functions in the sense of Cauchy-Fueter and has elegant applications to the functional calculus for non-commutative operators [11] and to Schur analysis [3]. For the detailed up-to-date theory, we refer to the monographs [20, 11]. The theory of slice regular functions centers around the non-elliptic differential operator with nonconstant coefficients, given by

$$|\operatorname{Im}(q)|^2 \frac{\partial}{\partial x_0} + \operatorname{Im}(q) \sum_{j=1}^3 x_j \frac{\partial}{\partial x_j},$$

where $\operatorname{Im}(q)$ is the imaginary part of the quaternion $q = x_0 + \operatorname{Im}(q) \in \mathbb{H}$; see [13] for more details. Furthermore, the notion of slice regularity was also extended to functions of an octonionic variable [23] and to the setting of Clifford algebras [12, 10] as well as to the setting of alternative real algebras [17].

The geometric theory of slice regular functions of one quaternionic variable has been already developed such as the Bloch-Landau theorem [15], the Bohr theorem [16], and the Landau-Toeplitz theorem [18]. Recently, the authors established the growth and distortion theorems for slice regular extensions of normalized univalent holomorphic functions with the tool of a so-called convex combination identity [30], and set up the Borel-Carathéodory theorems for slice regular functions using the method of finite average [29]. In the present article, we continue to investigate the geometric properties of slice regular functions and our starting point is the counterpart of Schwarz-Pick theorem in the setting of quaternions, which is first established in [5]; see [2] for an alternative proof.

Our main results in this article are the quaternionic versions of the Julia lemma, the Julia-Carathéodory theorem, the Hopf lemma, the Burns-Krantz rigidity theorem as well as the boundary Schwarz lemma for slice regular self-mappings.

Before presenting our main results, we first recall some notations. Let \mathbb{B} the open unit ball in the quaternions \mathbb{H} . For any $k > 0$ and any point $p \in \partial\mathbb{B}$, we denote

$$\mathcal{S}(p, k) = \{q \in \mathbb{B} : |p - q|^2 < k(1 - |q|^2)\}.$$

It is known that $\mathcal{S}(p, k)$ is an open ball internally tangent to the unit sphere $\partial\mathbb{B}$ with center $\frac{p}{1+k}$ and radius $\frac{k}{1+k}$. The boundary sphere of this ball is called an *orisphere*. These orispheres are crucial in the geometry theory concerning boundary behaviors of slice regular self-mappings of the open unit ball \mathbb{B} as shown in the following quaternionic counterpart of Julia's lemma.

Theorem 1. (Julia) *Let f be a slice regular self-mapping of the open unit ball \mathbb{B} and let $\xi \in \partial\mathbb{B}$. Suppose that there exists a sequence $\{q_n\}_{n \in \mathbb{N}} \subset \mathbb{B}$ converging to ξ as n tends to ∞ , such that the limits*

$$\alpha := \lim_{n \rightarrow \infty} \frac{1 - |f(q_n)|}{1 - |q_n|}$$

and

$$\eta := \lim_{n \rightarrow \infty} f(q_n)$$

exist (finitely). Then $\alpha > 0$ and the inequality

$$(1) \quad \operatorname{Re} \left((1 - f(q)\bar{\eta})^{-*} (1 + f(q)\bar{\eta}) \right) \geq \frac{1}{\alpha} \operatorname{Re} \left((1 - q\bar{\xi})^{-*} (1 + q\bar{\xi}) \right)$$

holds throughout the open unit ball \mathbb{B} and is strict except for regular Möbius transformations of \mathbb{B} .

In particular, the inequality (1) is equivalent to

$$(2) \quad \frac{|\eta - f(q)|^2}{1 - |f(q)|^2} \leq \alpha \frac{|1 - q|^2}{1 - |q|^2},$$

whenever $\xi = 1$. In other words,

$$f(\mathcal{S}(1, k)) \subseteq \mathcal{S}(\eta, \alpha k), \quad \forall k > 0.$$

Inequality (1) is called Julia's inequality in view of (2). It results in the quaternionic version of the Julia-Carathéodory theorem.

Theorem 2. (Julia-Carathéodory) *Let f be a slice regular self-mapping of the open unit ball \mathbb{B} . Then the following conditions are equivalent:*

(i) *The lower limit*

$$(3) \quad \alpha := \liminf_{q \rightarrow 1} \frac{1 - |f(q)|}{1 - |q|}$$

is finite, where the limit is taken as q approaches 1 unrestrictedly in \mathbb{B} ;

(ii) *f has a non-tangential limit, say η , at the point 1, and the difference quotient*

$$(q - 1)^{-1} (f(q) - \eta)$$

has a non-tangential limit, say $f'(1)$;

(iii) *The derivative f' has a non-tangential limit, say $f'(1)$, at the point 1.*

Moreover, under the above conditions we have

- (a) $\alpha > 0$ in (i);
- (b) the derivatives $f'(1)$ in (ii) and (iii) are the same;
- (c) $f'(1) = \alpha\eta$;
- (d) the quotient $\frac{1 - |f(q)|}{1 - |q|}$ has the non-tangential limit α .

It is worth remarking here that $f'(1)$ is closely related to α when α is finite. However, when $\alpha = \infty$, $f'(1)$ can be any quaternion β as demonstrated by the regular polynomial $f(q) = q^n \beta / n$ with $n > |\beta|$.

The quaternionic right half-space version of the Julia-Carathéodory theorem can also be established and its proof depends ultimately on the right half-space version of the Schwarz-Pick theorem (see Sect. 4). As a direct consequence, we obtain the Burns-Krantz rigidity theorem for slice regular functions with values in the closed right half-space

$$\overline{\mathbb{H}}^+ = \{q \in \mathbb{H} : \operatorname{Re}(q) \geq 0\};$$

see also [28, 24].

Theorem 3. (Burns-Krantz) *If $f : \mathbb{B} \rightarrow \overline{\mathbb{H}}^+$ is slice regular and*

$$f(q) = o(|q + 1|), \quad q \rightarrow -1,$$

then $f \equiv 0$.

The classical boundary Schwarz lemma claims that

$$(4) \quad \overline{\xi} f(\xi) \overline{f'(\xi)} \geq 1.$$

for any holomorphic function f on $\mathbb{D} \cup \{\xi\}$ satisfying $f(\mathbb{D}) \subseteq \mathbb{D}$, $f(0) = 0$ and $f(\xi) \in \partial\mathbb{D}$ for some point $\xi \in \partial\mathbb{D}$. However, its quaternionic variant has an additional item in terms of Lie bracket reflecting the non-commutativity of quaternions.

Theorem 4. (Schwarz) *Let $\xi \in \partial\mathbb{B}$ and f be a regular function on $\mathbb{B} \cup [\xi]$ such that $f(\mathbb{B}) \subseteq \mathbb{B}$ and $f(\xi) \in \partial\mathbb{B}$. Then:*

(i) *It holds that*

$$\overline{\xi} \left(f(\xi) \overline{f'(\xi)} + [\bar{\xi}, f(\xi) \overline{R_{\bar{\xi}} R_{\xi} f(\xi)}] \right) \geq \frac{1 - |f(0)|}{1 + |f(0)|},$$

where

$$R_{\xi} f(q) := (q - \xi)^{-*} * (f(q) - f(\xi)).$$

(ii) *If further*

$$f^{(k)}(0) = 0, \quad \forall k = 0, 1, \dots, n-1$$

for some $n \in \mathbb{N}$, then

$$\overline{\xi} \left(f(\xi) \overline{f'(\xi)} + [\bar{\xi}, f(\xi) \overline{R_{\bar{\xi}} R_{\xi} f(\xi)}] \right) \geq n + \frac{n! - |f^{(n)}(0)|}{n! + |f^{(n)}(0)|}.$$

In particular,

$$\overline{\xi} \left(f(\xi) \overline{f'(\xi)} + [\bar{\xi}, f(\xi) \overline{R_{\bar{\xi}} R_{\xi} f(\xi)}] \right) > n$$

unless $f(q) = q^n u$ for some $u \in \partial\mathbb{B}$.

Corollary 1. *Let f be a regular function on $\mathbb{B} \cup [\xi]$ such that $f(\mathbb{B}) \subseteq \mathbb{B}$, $f(0) = 0$ and $f(\xi) = \xi \in \partial\mathbb{B}$. Then*

$$f'(\xi) - [\xi, R_{\bar{\xi}} R_{\xi} f(\xi)] > 1$$

unless $f(q) = q$ for all $q \in \mathbb{B}$.

It is worth remarking here that the Lie bracket in the preceding corollary does not vanish and $f'(\xi)$ may not be a real number, in general; see Example 2 in Section 3 for more details. This means that in the setting of quaternions the commonly believed fact that

$$f'(\xi) > 1$$

may fail; see [20, Theorem 9.24].

An improved version of Theorem 4 for $\xi = 1$ is called Hopf's lemma, whose proof is based on the Lindelöf inequality.

Theorem 5. (Hopf) *Let f be a slice regular function on $\mathbb{B} \cup \{1\}$ such that $f(\mathbb{B}) \subseteq \mathbb{B}$ and $f(1) = 1$. Then the following statements hold true:*

(i) *The derivative of f at 1 is real and*

$$f'(1) \geq \frac{2|1 - f(0)|^2}{1 - |f(0)|^2 + |f'(0)|}.$$

In particular,

$$f'(1) \geq \frac{|1 - f(0)|^2}{1 - |f(0)|^2}.$$

Moreover, equality holds for the last inequality if and only if

$$f(q) = (1 - q\bar{u})^{-*} * (q - u)(1 - \bar{u})(1 - u)^{-1}$$

for some point $u \in \mathbb{B}$.

(ii) *If further*

$$f^{(k)}(0) = 0, \quad \forall k = 0, 1, \dots, n-1$$

for some $n \in \mathbb{N}$, then

$$f'(1) \geq n + \frac{n! - |f^{(n)}(0)|}{n! + |f^{(n)}(0)|}.$$

In particular, $f'(1) > n$ unless $f(q) = q^n$ for all $q \in \mathbb{B}$.

Notice that the Julia-Carathéodory theorem in Theorem 2 holds only for the real boundary points $\xi = \pm 1 \in \partial\mathbb{B}$. As shown by Theorem 4, the relation

$$f'(\xi) = \alpha \bar{\xi} f(\xi)$$

does no longer hold in general in the setting of quaternions under the condition that

$$\alpha := \liminf_{\mathbb{B} \ni q \rightarrow \xi} \frac{1 - |f(q)|}{1 - |q|} < +\infty$$

in contrast to the complex setting.

2. PRELIMINARIES

We recall in this section some preliminary definitions and results on slice regular functions. To have a more complete insight on the theory, we refer the reader to [20].

Let \mathbb{H} denote the non-commutative, associative, real algebra of quaternions with standard basis $\{1, i, j, k\}$, subject to the multiplication rules

$$i^2 = j^2 = k^2 = ijk = -1.$$

Every element $q = x_0 + x_1i + x_2j + x_3k$ in \mathbb{H} is composed by the *real* part $\operatorname{Re}(q) = x_0$ and the *imaginary* part $\operatorname{Im}(q) = x_1i + x_2j + x_3k$. The *conjugate* of $q \in \mathbb{H}$ is then $\bar{q} = \operatorname{Re}(q) - \operatorname{Im}(q)$ and its *modulus* is defined by $|q|^2 = q\bar{q} = |\operatorname{Re}(q)|^2 + |\operatorname{Im}(q)|^2$. We can therefore calculate the multiplicative inverse of each $q \neq 0$ as $q^{-1} = |q|^{-2}\bar{q}$. Every $q \in \mathbb{H}$ can be expressed as $q = x + yI$, where $x, y \in \mathbb{R}$ and

$$I = \frac{\operatorname{Im}(q)}{|\operatorname{Im}(q)|}$$

if $\operatorname{Im} q \neq 0$, otherwise we take I arbitrarily such that $I^2 = -1$. Then I is an element of the unit 2-sphere of purely imaginary quaternions

$$\mathbb{S} = \{q \in \mathbb{H} : q^2 = -1\}.$$

For every $I \in \mathbb{S}$ we will denote by \mathbb{C}_I the plane $\mathbb{R} \oplus I\mathbb{R}$, isomorphic to \mathbb{C} , and, if $\Omega \subseteq \mathbb{H}$, by Ω_I the intersection $\Omega \cap \mathbb{C}_I$. Also, for $R > 0$, we will denote the open ball centred at the origin with radius R by

$$B(0, R) = \{q \in \mathbb{H} : |q| < R\}.$$

We can now recall the definition of slice regularity.

Definition 1. Let Ω be a domain in \mathbb{H} . A function $f : \Omega \rightarrow \mathbb{H}$ is called *slice regular* if, for all $I \in \mathbb{S}$, its restriction f_I to Ω_I is *holomorphic*, i.e., it has continuous partial derivatives and satisfies

$$\bar{\partial}_I f(x + yI) := \frac{1}{2} \left(\frac{\partial}{\partial x} + I \frac{\partial}{\partial y} \right) f_I(x + yI) = 0$$

for all $x + yI \in \Omega_I$.

As shown in [9], the natural domains of definition of slice regular functions are the so-called axially symmetric slice domains.

Definition 2. Let Ω be a domain in \mathbb{H} . Ω is called a *slice domain* if Ω intersects the real axis and Ω_I is a domain of \mathbb{C}_I for any $I \in \mathbb{S}$.

Moreover, if $x + yI \in \Omega$ implies $x + y\mathbb{S} \subseteq \Omega$ for any $x, y \in \mathbb{R}$ and $I \in \mathbb{S}$, then Ω is called an *axially symmetric slice domain*.

From now on, we will omit the term ‘slice’ when referring to slice regular functions and will focus mainly on regular functions on $B(0, R) = \{q \in \mathbb{H} : |q| < R\}$ and the right half-space $\mathbb{H}^+ = \{q \in \mathbb{H} : \operatorname{Re}(q) > 0\}$. For regular functions the natural definition of derivative is given by the following (see [21, 22]).

Definition 3. Let $f : B(0, R) \rightarrow \mathbb{H}$ be a regular function. The *slice derivative* of f at $q = x + yI$ is defined by

$$\partial_I f(x + yI) := \frac{1}{2} \left(\frac{\partial}{\partial x} - I \frac{\partial}{\partial y} \right) f_I(x + yI).$$

Notice that the operators ∂_I and $\bar{\partial}_I$ commute, and $\partial_I f = \frac{\partial f}{\partial x}$ for regular functions. Therefore, the slice derivative of a regular function is still regular so that we can iterate the differentiation to obtain the n -th slice derivative

$$\partial_I^n f = \frac{\partial^n f}{\partial x^n}, \quad \forall n \in \mathbb{N}.$$

In what follows, for the sake of simplicity, we will denote the n -th slice derivative $\partial_I^n f$ by $f^{(n)}$ for every $n \in \mathbb{N}$.

In the theory of regular functions, the following splitting lemma (see [22]) relates closely slice regularity to classical holomorphy.

Lemma 1. (Splitting Lemma) *Let f be a regular function on $B = B(0, R)$. Then for any $I \in \mathbb{S}$ and any $J \in \mathbb{S}$ with $J \perp I$, there exist two holomorphic functions $F, G : B_I \rightarrow \mathbb{C}_I$ such that*

$$f_I(z) = F(z) + G(z)J, \quad \forall z = x + yI \in B_I.$$

Since the regularity does not keep under point-wise product of two regular functions a new multiplication operation, called the regular product (or $*$ -product), appears via a suitable modification of the usual one subject to noncommutative

setting. The regular product plays a key role in the theory of slice regular functions. On open balls centred at the origin, the $*$ -product of two regular functions is defined by means of their power series expansions (see, e.g., [19, 9]).

Definition 4. Let $f, g : B = B(0, R) \rightarrow \mathbb{H}$ be two regular functions and let

$$f(q) = \sum_{n=0}^{\infty} q^n a_n, \quad g(q) = \sum_{n=0}^{\infty} q^n b_n$$

be their series expansions. The regular product (or $*$ -product) of f and g is the function defined by

$$f * g(q) = \sum_{n=0}^{\infty} q^n \left(\sum_{k=0}^n a_k b_{n-k} \right)$$

and it is regular on B .

Notice that the $*$ -product is associative and is not, in general, commutative. Its connection with the usual pointwise product is clarified by the following result [19, 9].

Proposition 1. *Let f and g be regular on $B = B(0, R)$. Then for all $q \in B$,*

$$f * g(q) = \begin{cases} f(q)g(f(q)^{-1}qf(q)) & \text{if } f(q) \neq 0; \\ 0 & \text{if } f(q) = 0. \end{cases}$$

We remark that if $q = x + yI$ and $f(q) \neq 0$, then $f(q)^{-1}qf(q)$ has the same modulus and same real part as q . Therefore $f(q)^{-1}qf(q)$ lies in the same 2-sphere $x + y\mathbb{S}$ as q . Notice that a zero $x_0 + y_0I$ of the function g is not necessarily a zero of $f * g$, but some element on the same sphere $x_0 + y_0\mathbb{S}$ does. In particular, a real zero of g is still a zero of $f * g$. To present a characterization of the structure of the zero set of a regular function f , we need to introduce the following functions.

Definition 5. Let $f(q) = \sum_{n=0}^{\infty} q^n a_n$ be a regular function on $B = B(0, R)$. We define the *regular conjugate* of f as

$$f^c(q) = \sum_{n=0}^{\infty} q^n \bar{a}_n,$$

and the *symmetrization* of f as

$$f^s(q) = f * f^c(q) = f^c * f(q) = \sum_{n=0}^{\infty} q^n \left(\sum_{k=0}^n a_k \bar{a}_{n-k} \right).$$

Both f^c and f^s are regular functions on B .

We are now able to define the inverse element of a regular function f with respect to the $*$ -product. Let \mathcal{Z}_{f^s} denote the zero set of the symmetrization f^s of f .

Definition 6. Let f be a regular function on $B = B(0, R)$. If f does not vanish identically, its *regular reciprocal* is the function defined by

$$f^{-*}(q) := f^s(q)^{-1} f^c(q)$$

and it is regular on $B \setminus \mathcal{Z}_{f^s}$.

The following result shows that the regular quotient is nicely related to the pointwise quotient (see [36, 37]).

Proposition 2. *Let f and g be regular on $B = B(0, R)$. Then for all $q \in B \setminus \mathcal{Z}_{f^s}$,*

$$f^{-*} * g(q) = f(T_f(q))^{-1} g(T_f(q)),$$

where $T_f : B \setminus \mathcal{Z}_{f^s} \rightarrow B \setminus \mathcal{Z}_{f^s}$ is defined by $T_f(q) = f^c(q)^{-1} q f^c(q)$. Furthermore, T_f and T_{f^c} are mutual inverses so that T_f is a diffeomorphism.

Let us set

$$U(x_0 + y_0\mathbb{S}, R) = \{q \in \mathbb{H} : |(q - x_0)^2 + y_0^2| < R^2\}$$

for all $x_0, y_0 \in \mathbb{R}$ and all $r > 0$. The following result was proved in [38]; see Theorems 4.1 and 6.1 there for more details.

Theorem 6. *Let f be a regular function on a symmetric slice domain Ω , and let $q_0 = x_0 + Iy_0 \in U(x_0 + y_0\mathbb{S}, R) \subseteq \Omega$. Then there exists $\{A_n\}_{n \in \mathbb{N}} \subset \mathbb{H}$ such that*

$$(5) \quad f(q) = \sum_{n=0}^{\infty} ((q - x_0)^2 + y_0^2)^n (A_{2n} + (q - q_0)A_{2n+1})$$

for all $q \in U(x_0 + y_0\mathbb{S}, R)$.

As a consequence, for all $v \in \mathbb{H}$ with $|v| = 1$ the directional derivative of f along v can be computed at q_0 as

$$\frac{\partial f}{\partial v}(q_0) = \lim_{t \rightarrow 0} \frac{f(q_0 + tv) - f(q_0)}{t} = vA_1 + (q_0v - v\overline{q_0})A_2,$$

where

$$A_1 = R_{q_0}f(\overline{q_0}) = \partial_s f(q_0), \quad A_2 = R_{\overline{q_0}}R_{q_0}f(q_0).$$

In particular, there holds that

$$f'(q_0) = R_{q_0}f(q_0) = A_1 + 2\operatorname{Im}(q_0)A_2.$$

3. PROOFS OF MAIN THEOREMS

In this section, we shall give the proofs of Theorems 1-5 except that of Theorem 3, which will be given in the next section.

Proof of Theorem 1. The Schwarz-Pick Theorem shows, for all $q \in \mathbb{B}$,

$$|(1 - f(q)\overline{f(q_n)})^{-*} * (f(q) - f(q_n))| \leq |(1 - q\overline{q_n})^{-*} * (q - \overline{q_n})|,$$

which together with Proposition 2 implies that

$$\frac{|f \circ T_{1-f\overline{f(q_n)}}(q) - f(q_n)|}{|1 - f \circ T_{1-f\overline{f(q_n)}}(q)\overline{f(q_n)}|} \leq \frac{|T_{1-I d\overline{q_n}}(q) - \overline{q_n}|}{|1 - T_{1-I d\overline{q_n}}(q)\overline{q_n}|}.$$

We square both sides and then minus by one to yield

$$\frac{|1 - f \circ T_{1-f\overline{f(q_n)}}(q)\overline{f(q_n)}|^2}{1 - |f \circ T_{1-f\overline{f(q_n)}}(q)|^2} \leq \frac{|1 - T_{1-I d\overline{q_n}}(q)\overline{q_n}|^2}{1 - |q|^2} \frac{1 - |f(q_n)|^2}{1 - |q_n|^2}.$$

Letting $n \rightarrow \infty$, we obtain that

$$(6) \quad \frac{|1 - f \circ T_{1-f\overline{\eta}}(q)\overline{\eta}|^2}{1 - |f \circ T_{1-f\overline{\eta}}(q)|^2} \leq \alpha \frac{|1 - T_{1-I d\overline{\xi}}(q)\overline{\xi}|^2}{1 - |q|^2}.$$

It implies that $\alpha > 0$ and

$$\frac{1 - |f \circ T_{1-f\bar{\eta}}(q)|^2}{|1 - f \circ T_{1-f\bar{\eta}}(q)\bar{\eta}|^2} \geq \frac{1}{\alpha} \frac{1 - |q|^2}{|1 - T_{1-I d\bar{\xi}}(q)\bar{\xi}|^2}.$$

This is equivalent to the inequality

$$\operatorname{Re}\left((1 - f \circ T_{1-f\bar{\eta}}(q)\bar{\eta})^{-1}(1 + f \circ T_{1-f\bar{\eta}}(q)\bar{\eta})\right) \geq \frac{1}{\alpha} \operatorname{Re}\left((1 - T_{1-I d\bar{\xi}}(q)\bar{\xi})^{-1}(1 + T_{1-I d\bar{\xi}}(q)\bar{\xi})\right).$$

That is, in terms of the regular product,

$$\operatorname{Re}\left((1 - f(q)\bar{\eta})^{-*} * (1 + f(q)\bar{\eta})\right) \geq \frac{1}{\alpha} \operatorname{Re}\left((1 - q\bar{\xi})^{-*} * (1 + q\bar{\xi})\right).$$

In particular, when $\xi = 1$, inequality (6) becomes

$$(7) \quad \frac{|1 - f \circ T_{1-f\bar{\eta}}(q)\bar{\eta}|^2}{|1 - |f \circ T_{1-f\bar{\eta}}(q)|^2} \leq \alpha \frac{|1 - q|^2}{1 - |q|^2}$$

due to the fact that

$$|1 - T_{1-I d}(q)| = |1 - q|.$$

By Proposition 2, $T_{1-f\bar{\eta}}$ is a homeomorphism with inverse $T_{1-\eta*f^c}$ since $f(\mathbb{B}) \subseteq \mathbb{B}$. Replacing q by $T_{1-\eta*f^c}(q)$ in inequality (7) gives that

$$\frac{|\eta - f(q)|^2}{|1 - f(q)|^2} \leq \alpha \frac{|1 - T_{1-\eta*f^c}(q)|^2}{1 - |q|^2} = \alpha \frac{|1 - q|^2}{1 - |q|^2}, \quad \forall q \in \mathbb{B}.$$

If equality holds for Julia's inequality (1) at some point $q_0 \in \mathbb{B}$, then the function

$$(1 - f(q)\bar{\eta})^{-*} * (1 + f(q)\bar{\eta}) - \frac{1}{\alpha}(1 - q\bar{\xi})^{-*} * (1 + q\bar{\xi})$$

is an imaginary constant, say It_0 in virtue of the maximum principle for real part of regular functions (see Lemma 2 in [29]). A simple calculation shows that

$$f(q) = \left(1 + \frac{1}{\alpha}(1 - q\bar{\xi})^{-*} * (1 + q\bar{\xi}) + It_0\right)^{-*} * \left(\frac{1}{\alpha}(1 - q\bar{\xi})^{-*} * (1 + q\bar{\xi}) + It_0 - 1\right)\eta.$$

Notice that the term in the first bracket can be written as

$$\frac{1}{\alpha}(1 - q\bar{\xi})^{-*} * (1 + \alpha + I\alpha t_0) * (1 + q\bar{u}),$$

where

$$u = ((1 - \alpha) + I\alpha t_0)\xi((1 + \alpha) - I\alpha t_0)^{-1} \in \mathbb{B},$$

since $\alpha > 0$. The other term can be treated similarly. Consequently, f can be represented as

$$f(q) = (1 + q\bar{u})^{-*} * (q + u)v,$$

where

$$v = ((1 + \alpha) + I\alpha t_0)^{-1}\bar{\xi}((1 + \alpha) - I\alpha t_0)\eta \in \partial\mathbb{B}.$$

It follows that the equality in Julia's inequality can hold only for regular Möbius transformations of \mathbb{B} onto \mathbb{B} , and a direct calculation shows that it does indeed hold for all such regular Möbius transformations. Now the proof is complete. \square

To prove Theorem 2, we shall need a quaternionic version of Lindelöf's principle, which follows easily from the corresponding result in the complex setting and the splitting lemma.

Lemma 2. (Lindelöf) *Let f be a regular function on \mathbb{B} and bounded in each non-tangential approach region at 1. If for some continuous curve $\gamma \in \mathbb{B} \cap \mathbb{C}_I$ ending at 1 for some $I \in \mathbb{S}$, there exists the limit*

$$\eta = \lim_{t \rightarrow 1^-} f(\gamma(t)),$$

then f also has the non-tangential limit η at 1.

Now we are in a position to prove the Julia-Carathéodory theorem.

Proof of Theorem 2. The equivalence between (ii) and (iii) follows directly from the corresponding result in the complex setting and the splitting lemma.

The implication (ii) \Rightarrow (i) follows from the inequality

$$\frac{1 - |f(r)|}{1 - r} \leq \frac{|\eta - f(r)|}{1 - r}, \quad \forall r \in (0, 1).$$

Now we prove the implication (i) \Rightarrow (ii). Under assumption (i), there exists a sequence $\{q_n\}_{n \in \mathbb{N}} \subset \mathbb{B}$ converging to 1 as n tends to ∞ , such that

$$\alpha = \lim_{n \rightarrow \infty} \frac{1 - |f(q_n)|}{1 - |q_n|}$$

and

$$\lim_{n \rightarrow \infty} f(q_n) = \eta$$

for some $\eta \in \partial\mathbb{B}$. It follows from Julia's inequality (2) that

$$\frac{|\eta - f(q)|^2}{1 - |f(q)|^2} \leq \alpha \frac{|1 - q|^2}{1 - |q|^2}.$$

Fix a non-tangential approach region at 1, say the region

$$(8) \quad \mathcal{R}(1, k) = \{q \in \mathbb{B} : |q - 1| < k(1 - |q|)\},$$

where k is a constant greater than one. For all $q \in \mathcal{R}(1, k)$ we have

$$(9) \quad \frac{|\eta - f(q)|^2}{1 - |f(q)|^2} \leq \alpha k |q - 1| \frac{(1 - |q|)}{1 - |q|^2} \leq \alpha k |q - 1|,$$

which implies that $f(q)$ tends to η as q tends to 1 within $\mathcal{R}(1, k)$. In other words, f has a non-tangential limit η at 1.

It remains to prove that the difference quotient

$$(q - 1)^{-1}(f(q) - \eta)$$

has a non-tangential limit $\alpha\eta$. To this end, notice that

$$\frac{|\eta - f(q)|}{1 + |f(q)|} \leq \frac{|\eta - f(q)|^2}{1 - |f(q)|^2},$$

from which and inequality (9) we have

$$|(q - 1)^{-1}(f(q) - \eta)| \leq \alpha k (1 + |f(q)|) \leq 2\alpha k,$$

whenever $q \in \mathcal{R}(1, k)$. Consequently, the regular function

$$g(q) := (q - 1)^{-1}(f(q) - \eta)$$

is bounded in each non-tangential approach region at 1. The Lindelöf's principle in Lemma 2 thus reduces the proof to the existence of the radial limit

$$\lim_{r \rightarrow 1^-} \frac{\eta - f(r)}{1 - r} = \alpha\eta.$$

To consider this radial limit, we observe from the definition of α as the lower limit in (i) that

$$(10) \quad \liminf_{r \rightarrow 1^-} \frac{1 - |f(r)|}{1 - r} \geq \alpha.$$

On the other hand, setting $q = r$ in the Julia's inequality (2) yields that

$$\frac{1 - |f(r)|}{1 - r} \frac{1 + r}{1 + |f(r)|} \leq \frac{|\eta - f(r)|^2}{1 - |f(r)|^2} \frac{1 + r}{1 - r} \leq \alpha \frac{(1 - r)^2}{1 - r^2} \frac{1 + r}{1 - r}.$$

Since f has a non-tangential limit $\eta \in \partial\mathbb{B}$ at 1, it follows that

$$(11) \quad \limsup_{r \rightarrow 1^-} \frac{1 - |f(r)|}{1 - r} \leq \alpha.$$

Consequently,

$$(12) \quad \lim_{r \rightarrow 1^-} \frac{1 - |f(r)|}{1 - r} = \alpha.$$

Furthermore,

$$\left(\frac{1 - |f(r)|}{1 - r} \right)^2 \leq \left(\frac{|\eta - f(r)|}{1 - r} \right)^2 \leq \alpha \frac{1 - |f(r)|^2}{1 - r^2} = \alpha \frac{1 - |f(r)|}{1 - r} \frac{1 + |f(r)|}{1 + r},$$

which together with (12) implies that

$$(13) \quad \lim_{r \rightarrow 1^-} \frac{|\eta - f(r)|}{1 - r} = \alpha.$$

By (12) and (13), we have

$$(14) \quad \lim_{r \rightarrow 1^-} \frac{1 - |f(r)|}{|\eta - f(r)|} = 1.$$

Since

$$\frac{1 - |f(r)|}{|\eta - f(r)|} \leq \frac{1 - \operatorname{Re}(f(r)\bar{\eta})}{|1 - f(r)\bar{\eta}|} \leq 1,$$

it follows from that

$$\lim_{r \rightarrow 1^-} \operatorname{Re} \frac{1 - f(r)\bar{\eta}}{|1 - f(r)\bar{\eta}|} = 1.$$

This forces that

$$\lim_{r \rightarrow 1^-} \frac{1 - f(r)\bar{\eta}}{|1 - f(r)\bar{\eta}|} = 1.$$

Therefore,

$$(15) \quad \begin{aligned} \lim_{r \rightarrow 1^-} \frac{\eta - f(r)}{1 - r} &= \lim_{r \rightarrow 1^-} \frac{1 - f(r)\bar{\eta}}{1 - r} \eta \\ &= \lim_{r \rightarrow 1^-} \frac{1 - f(r)\bar{\eta}}{|1 - f(r)\bar{\eta}|} \frac{1 - |f(r)|}{1 - r} \frac{|\eta - f(r)|}{1 - |f(r)|} \eta \\ &= \alpha \eta, \end{aligned}$$

which completes the proof of the implication (i) \Rightarrow (ii).

Now we assume that all conditions (i)-(iii) hold. By carefully checking the above proof, we see that assertions (a)-(c) hold true. It remains to verify (d).

From assertions (i), (ii), and (c), we know that $\alpha < \infty$ and the difference quotient

$$(q - 1)^{-1}(f(q) - \eta)$$

has a non-tangential limit $\alpha\eta$ at point 1. Let us set

$$g(q) = (q-1)^{-1}(f(q)\bar{\eta} - 1) - \alpha.$$

Then g is regular on \mathbb{B} and has the non-tangential limit 0 at point 1. Since

$$|f(q)|^2 = |1 + (q-1)(\alpha + g(q))|^2,$$

it follows that

$$(16) \quad \frac{1 - |f(q)|^2}{1 - |q|^2} = 2\alpha \frac{\operatorname{Re}(1-q)}{1 - |q|^2} + 2 \frac{\operatorname{Re}((1-q)g(q))}{1 - |q|^2} - \frac{|1-q|^2}{1 - |q|^2} |\alpha + g(q)|^2.$$

Fix the non-tangential approach region $\mathcal{R}(1, k)$ as in (8) and consider the non-tangential limit as $q \rightarrow 1$ within $\mathcal{R}(1, k)$. It is clear that the second term on the right-hand side of the preceding equality approaches 0 and so does the third term. Since the equality

$$\frac{\operatorname{Re}(1-q)}{1 - |q|^2} = \frac{1}{2} \left(1 + \frac{|1-q|^2}{1 - |q|^2} \right)$$

shows that

$$\lim_{\substack{q \rightarrow 1 \\ q \in \mathcal{R}(1, k)}} \frac{\operatorname{Re}(1-q)}{1 - |q|^2} = \frac{1}{2},$$

from which it follows that the first term on the right-hand side of equality (16) approaches to α as $q \rightarrow 1$ within $\mathcal{R}(1, k)$. Therefore,

$$\lim_{\substack{q \rightarrow 1 \\ q \in \mathcal{R}(1, k)}} \frac{1 - |f(q)|^2}{1 - |q|^2} = \alpha,$$

which is equivalent to

$$\lim_{\substack{q \rightarrow 1 \\ q \in \mathcal{R}(1, k)}} \frac{1 - |f(q)|}{1 - |q|} = \alpha.$$

Now the proof is complete. \square

To prove Hopf's theorem, we need the following Lindelöf inequality.

Proposition 3. (Lindelöf) *Let f be a slice regular self-mapping of the open unit ball \mathbb{B} . Then for all $q \in \mathbb{B}$ we have*

$$(17) \quad \left| f(q) - \frac{1 - |q|^2}{1 - |q|^2 |f(0)|^2} f(0) \right| \leq \frac{|q|(1 - |f(0)|^2)}{1 - |q|^2 |f(0)|^2};$$

$$(18) \quad \frac{|f(0)| - |q|}{1 - |q||f(0)|} \leq |f(q)| \leq \frac{|q| + |f(0)|}{1 + |q||f(0)|};$$

$$(19) \quad |f(q) - f(0)| \leq \frac{|q|(1 - |f(0)|^2)}{1 - |q||f(0)|}.$$

Moreover, if there exists a positive integer n such that

$$f^{(k)}(0) = 0, \quad k = 0, 1, \dots, n-1,$$

then it holds that

$$(20) \quad \frac{|f^{(n)}(0)| - n!|q|}{n! - |q||f^{(n)}(0)|} |q|^n \leq |f(q)| \leq \frac{n!|q| + |f^{(n)}(0)|}{n! + |q||f^{(n)}(0)|} |q|^n, \quad \forall q \in \mathbb{B}.$$

Equality holds for one of inequalities in (17) – (19) at some point $q_0 \in \mathbb{B} \setminus \{0\}$ if and only if f is a regular Möbius transformations of \mathbb{B} onto \mathbb{B} , and equality holds for one of inequalities in (20) at some point $q_0 \in \mathbb{B} \setminus \{0\}$ if and only if $f(q) = q^n u$ for some point $u \in \partial\mathbb{B}$.

Proof. The Schwarz-Pick Theorem gives, for all $q \in \mathbb{B}$,

$$(21) \quad |(1 - f(q)\overline{f(0)})^{-*} * (f(q) - f(0))| \leq |q|,$$

which together with Proposition 2 implies that

$$\frac{|f \circ T_{1-f\overline{f(0)}}(q) - f(0)|}{|1 - f \circ T_{1-f\overline{f(0)}}(q)\overline{f(0)}|} \leq |q|.$$

An easy calculation shows that the preceding inequality is equivalent to

$$\left| f \circ T_{1-f\overline{f(0)}}(q) - \frac{1 - |q|^2}{1 - |q|^2|f(0)|^2} f(0) \right| \leq \frac{|q|(1 - |f(0)|^2)}{1 - |q|^2|f(0)|^2}.$$

Since $f(\mathbb{B}) \subseteq \mathbb{B}$, it follows from Proposition 2 that $T_{1-f\overline{f(0)}}$ is a homeomorphism with inverse $T_{1-f(0)*f^c}$. Replacing q by $T_{1-f(0)*f^c}(q)$ in the preceding inequality gives that

$$\left| f(q) - \frac{1 - |q|^2}{1 - |q|^2|f(0)|^2} f(0) \right| \leq \frac{|q|(1 - |f(0)|^2)}{1 - |q|^2|f(0)|^2}, \quad \forall q \in \mathbb{B}.$$

If equality achieves at some point $0 \neq q_0 \in \mathbb{B}$ in the last inequality, then it also achieves at point $0 \neq \tilde{q}_0 = T_{1-f(0)*f^c}(q_0)$ in inequality (21). It thus follows from the Schwarz-Pick Theorem that f is a regular Möbius transformations of \mathbb{B} onto \mathbb{B} .

Inequalities in (18) and (19) follow from (17) as well as the triangle inequality. Moreover, if $f^{(k)}(0) = 0$, $k = 0, 1, \dots, n-1$ for some positive integer n , then the function $g(q) := q^{-n}f(q)$ is also a regular function from \mathbb{B} to $\overline{\mathbb{B}}$ in virtue of the maximum principle, and in addition we have $g(0) = f^{(n)}(0)/n!$. Therefore, inequalities in (20) follow from (18). The proof is complete. \square

We now come to prove the quaternionic version of Hopf's theorem.

Proof of Theorem 5. (ii) If $|f^{(n)}(0)| = n!$, then it follows from the Schwarz Lemma that $f(q) = q^n$ for all $q \in \mathbb{B}$, and hence the result holds. Otherwise, $|f^{(n)}(0)| < n!$ in virtue of the Cauchy inequality. By the Julia-Carathéodory Theorem, $f'(1) = \alpha \in (0, +\infty)$. Moreover, it follows from the second inequality in (20) that

$$(22) \quad \frac{1 - |f(q)|}{1 - |q|} \geq \frac{n!(1 - |q|^{n+1}) + |f^{(n)}(0)||q|(1 - |q|^{n-1})}{(1 - |q|)(n! + |q||f^{(n)}(0)|)}.$$

Letting $q \rightarrow 1$ within $\mathcal{R}(1, k)$, we have

$$f'(1) = \alpha \geq \frac{(n+1)! + (n-1)|f^{(n)}(0)|}{n! + |f^{(n)}(0)|} = n + \frac{n! - |f^{(n)}(0)|}{n! + |f^{(n)}(0)|} > n,$$

since $|f^{(n)}(0)| < n!$ by our assumption.

If $f'(1) = n$, then $|f^{(n)}(0)| = n!$. Consequently, the regular function $g(q) := q^{-n}f(q)$ which maps \mathbb{B} to $\overline{\mathbb{B}}$ attains its maximum in modulus at the point 0, and hence is a unimodular constant $u \in \partial\mathbb{B}$, in virtue of the maximum principle. Since $g(1) = f(1) = 1$, it follows from that $u = 1$ and $f(q) = q^n$ for all $q \in \mathbb{B}$.

(i) The result follows easily from (ii) by considering the regular function

$$(1 - f(q)\overline{f(0)})^{-*} * (f(q) - f(0))(1 - \overline{f(0)})(1 - f(0))^{-1},$$

and noticing

$$f'(1) = \frac{|1 - f(0)|^2}{1 - |f(0)|^2} g'(1), \quad |g'(0)| = \frac{|f'(0)|}{1 - |f(0)|^2} \leq 1.$$

Now the proof is complete. \square

To prove the boundary Schwarz lemma (Theorem 4), we make the best use of the classical Hopf's lemma, which can be viewed as the real version of the boundary Schwarz lemma. We remark that, unlike in the complex setting, the boundary Schwarz lemma can not be simplified to the specific case that $\xi = 1$ because the theory of regular composition is unavailable.

Proof of Theorem 4. By assumption, the function $|f|^2$ attains its maximum at the boundary point $\xi \in \partial\mathbb{B}$ so that the directional derivative of $|f|^2$ along ξ at the point ξ satisfies that

$$(23) \quad \frac{\partial |f|^2}{\partial \xi}(\xi) > 0,$$

in virtue of the classical Hopf's lemma. Moreover,

$$(24) \quad \frac{\partial |f|^2}{\partial \tau}(\xi) = 0, \quad \forall \tau \in T_\xi(\partial\mathbb{B}) \cong \mathbb{R}^3.$$

Indeed, for any unit tangent vector $\tau \in T_\xi(\partial\mathbb{B})$, take a smooth curve $\gamma : (-1, 1) \rightarrow \overline{\mathbb{B}}$ such that

$$\gamma(0) = \xi, \quad \gamma'(0) = \tau.$$

By definition we have

$$\frac{\partial |f|^2}{\partial \tau}(\xi) = \left(\frac{d}{dt} |f(\gamma(t))|^2 \right) \Big|_{t=0} = 0,$$

since the function $|f(\gamma(t))|^2$ in t attains its maximum at the point $t = 0$.

Moreover, Theorem 6 shows that

$$\frac{\partial f}{\partial v}(\xi) = vA_1 + (\xi v - v\bar{\xi})A_2$$

for all $v \in \mathbb{H}$ with $|v| = 1$. Therefore,

$$(25) \quad \begin{aligned} \frac{\partial |f|^2}{\partial v}(\xi) &= 2 \left\langle \frac{\partial f}{\partial v}(\xi), f(\xi) \right\rangle \\ &= 2 \left\langle vA_1 + (\xi v - v\bar{\xi})A_2, f(\xi) \right\rangle \\ &= 2 \left\langle v, f(\xi)\overline{A_1} + \bar{\xi}f(\xi)\overline{A_2} - f(\xi)\overline{A_2}\xi \right\rangle \\ &= 2 \left\langle v, f(\xi)\overline{f'(\xi)} + [\bar{\xi}, f(\xi)\overline{A_2}] \right\rangle, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{H} \cong \mathbb{R}^4$, i.e.,

$$\langle p, q \rangle = \operatorname{Re}(p\bar{q}).$$

In the last equality we have used the fact that

$$f'(\xi) = A_1 + 2\operatorname{Im}(\xi)A_2.$$

Now it follows from (24) and (25) that

$$f(\xi)\overline{f'(\xi)} + [\bar{\xi}, f(\xi)\overline{A_2}] \perp T_\xi(\partial\mathbb{B})$$

so that in view of (23) and (25) there exists a real number $\lambda > 0$ such that

$$f(\xi)\overline{f'(\xi)} + [\bar{\xi}, f(\xi)\overline{A_2}] = \lambda \xi$$

and

$$\bar{\xi} \left(f(\xi)\overline{f'(\xi)} + [\bar{\xi}, f(\xi)\overline{A_2}] \right) = \lambda = \frac{1}{2} \frac{\partial |f|^2}{\partial \xi}(\xi) = \frac{\partial |f|}{\partial \xi}(\xi) > 0.$$

In order to get a better lower bound for the directional derivative $\frac{\partial |f|}{\partial \xi}(\xi)$, we make the best use of the so-called Lindelöf's inequality. From the second inequality in (18), we have

$$\frac{1 - |f(q)|}{1 - |q|} \geq \frac{1 - |f(0)|}{1 + |q||f(0)|},$$

which implies that

$$\frac{\partial |f|}{\partial \xi}(\xi) = \lim_{r \rightarrow 1^-} \frac{1 - |f(r\xi)|}{1 - r} \geq \frac{1 - |f(0)|}{1 + |f(0)|}.$$

If further

$$f^{(k)}(0) = 0, \quad \forall k = 0, 1, \dots, n-1$$

for some $n \in \mathbb{N}$, then it follows from inequality (22) that

$$\frac{1 - |f(r\xi)|}{1 - r} \geq \frac{n!(1 - r^{n+1}) + r(1 - r^{n-1})|f^{(n)}(0)|}{(1 - r)(n! + r|f^{(n)}(0)|)}.$$

Consequently,

$$\frac{\partial |f|}{\partial \xi}(\xi) \geq \frac{(n+1)! + (n-1)|f^{(n)}(0)|}{n! + |f^{(n)}(0)|} = n + \frac{n! - |f^{(n)}(0)|}{n! + |f^{(n)}(0)|} \geq n,$$

and hence

$$\bar{\xi} \left(f(\xi)\overline{f'(\xi)} + [\bar{\xi}, f(\xi)\overline{A_2}] \right) = \frac{\partial |f|}{\partial \xi}(\xi) \geq n.$$

Furthermore, if equality holds for the last inequality, then $|f^{(n)}(0)| = n!$ so that $f(q) = q^n u$ for some unimodular constant $u \in \partial\mathbb{B}$. Now the proof is complete. \square

Remark 1. It is quite natural to ask if the quality

$$\bar{\xi} \left(f(\xi)\overline{f'(\xi)} + [\bar{\xi}, f(\xi)\overline{R_{\bar{\xi}}R_{\xi}f(\xi)}] \right)$$

in Theorem 4 is no other than

$$\bar{\xi} f(\xi)\overline{f'(\xi)}$$

as in the complex setting. Unfortunately, the Lie bracket

$$[\bar{\xi}, f(\xi)\overline{R_{\bar{\xi}}R_{\xi}f(\xi)}]$$

in Theorem 4 does not vanish in general. Moreover, all the products of ξ , $f(\xi)$ and $\overline{f'(\xi)}$ in any different orders may fail simultaneously to be real numbers so that the inequality

$$\bar{\xi} f(\xi)\overline{f'(\xi)} \geq \frac{1 - |f(0)|}{1 + |f(0)|}$$

does not hold, neither all of its modified versions free of orders. These facts can be demonstrated by the following counterexample.

Example 1. Let $I \in \mathbb{S}$ be fixed. Set

$$f(q) = (1 + qI/2)^{-*} * (q - I/2).$$

Then it is a slice regular Möbius transformation of \mathbb{B} onto \mathbb{B} . It is evident to see that

$$f(q) = (q^2 + 4)^{-1} (3q - 2(q^2 + 1)I)$$

so that it satisfies all the assumptions of Theorem 4.

Now we set $\xi = J$, where $J \in \mathbb{S}$ is fixed such that $J \perp I$. An easy calculation thus shows that

$$f(J) = J, \quad f'(J) = \frac{1}{3}(5 + 4IJ),$$

and

$$A_1 = \partial_s f(J) = 1, \quad A_2 = R_{-J} R_J f(J) = -\frac{1}{3}(2I + J).$$

Therefore,

$$[\overline{J}, f(J) \overline{R_{-J} R_J f(J)}] = -\frac{1}{3}[J, J(2I + J)] = \frac{4}{3}I \neq 0.$$

On the other hand,

$$\overline{J} \left(f(J) \overline{f'(J)} + [\overline{J}, f(J) \overline{R_{-J} R_J f(J)}] \right) = \frac{5}{3} > \frac{1 - |f(0)|}{1 + |f(0)|} = \frac{1}{3}$$

as predicated by Theorem 4.

Now we provide an example to show that in Corollary 1 the inequality $f'(\xi) > 1$ may fail, or rather that $f'(\xi)$ may not be a real number.

Example 2. We now construct a function g such that $g'(J)$ is no longer a real number. To this end, we set

$$g(q) = -qf(q)J = -q(1 + qI/2)^{-*} * (q - I/2)J,$$

where f is as described in Example 1 and $I, J \in \mathbb{S}$ with $J \perp I$.

It is evident that this function is a Blaschke product of order 2 so that it is regular on \mathbb{B} , and satisfies $g(\mathbb{B}) \subseteq \mathbb{B}$, $g(0) = 0$ and $g(J) = -Jf(J)J = J$. This means that g satisfies all assumptions given in Corollary 1.

However, we find that $g'(J)$ is indeed not a real number. In fact, by the Leibniz rule we have

$$g'(q) = -(f(q) + qf'(q))J.$$

Consequently,

$$g'(J) = \frac{4}{3}(2 - IJ) \notin \mathbb{R}.$$

On the other hand, a simple calculation shows that

$$[J, R_{-J} R_J g(J)] = -\frac{4}{3}IJ$$

and

$$g'(J) - [J, R_{-J} R_J g(J)] = \frac{8}{3} > 1$$

as predicated by Corollary 1.

4. JULIA-CARATHÉODORY THEOREM IN \mathbb{H}^+

In this section, we establish the Julia-Carathéodory theorem for slice regular self-mappings of the right half-space

$$\mathbb{H}^+ := \{q \in \mathbb{H} : \operatorname{Re}(q) > 0\}.$$

The proof depends ultimately on the right half-space version of the Schwarz-Pick theorem.

Theorem 7. (Schwarz-Pick) *Let $f : \mathbb{H}^+ \rightarrow \mathbb{H}^+$ be a regular function. Then for any $q_0 \in \mathbb{H}^+$ we have*

$$\left| (g(q) + \overline{g(q_0)})^{-*} * (g(q) - g(q_0)) \right| \leq \left| (q + \overline{q_0})^{-*} * (q - q_0) \right|, \quad \forall q \in \mathbb{H}^+.$$

Inequality is strict (except at $q = q_0$) unless f is a regular Möbius transformation from \mathbb{H}^+ onto itself.

We denote the non-tangential cone at the boundary point 0 of \mathbb{H}^+ as

$$\mathcal{S}_\gamma = \{q \in \mathbb{H}^+ : \operatorname{Re}(q) > \gamma|q|\}$$

for any $\gamma > 0$.

Theorem 8. (Julia-Carathéodory) *Let $f : \mathbb{H}^+ \rightarrow \mathbb{H}^+$ be a regular function and set*

$$c := \inf \left\{ \frac{\operatorname{Re} f(q)}{\operatorname{Re}(q)} : q \in \mathbb{H}^+ \right\} \geq 0.$$

Then the following hold true:

(i) *for any $q \in \mathbb{H}^+$,*

$$\operatorname{Re} f(q) \geq c \operatorname{Re}(q);$$

(ii) *for any $\gamma > 0$,*

$$\lim_{\substack{|q| \rightarrow \infty \\ q \in \mathcal{S}_\gamma}} q^{-1} f(q) = \lim_{\substack{|q| \rightarrow \infty \\ q \in \mathcal{S}_\gamma}} \frac{\operatorname{Re} f(q)}{\operatorname{Re}(q)} = c;$$

(iii) *for any $\gamma > 0$,*

$$\lim_{\substack{|q| \rightarrow \infty \\ q \in \mathcal{S}_\gamma}} f'(q) = c.$$

Proof. We put

$$(26) \quad g(q) := f(q) - cq, \quad \forall q \in \mathbb{H}^+,$$

Then by definition $\operatorname{Re} g(q) \geq 0$ for all $q \in \mathbb{H}^+$. Moreover, we may assume that

$$\operatorname{Re} g(q) > 0, \quad \forall q \in \mathbb{H}^+,$$

in virtue of the maximum principle for real parts of regular functions, see Lemma 2 in [29]. Otherwise, $g(q) = It_0$ for some $t_0 \in \mathbb{R}$ and some $I \in \mathbb{S}$. Thus the results are obvious.

It follows from the Schwarz-Pick theorem that for all $q, q_0 \in \mathbb{H}^+$,

$$\left| (g(q) + \overline{g(q_0)})^{-*} * (g(q) - g(q_0)) \right| \leq \left| (q + \overline{q_0})^{-*} * (q - q_0) \right|,$$

which is equivalent to

$$\frac{|g \circ T_{g+\overline{g(q_0)}}(q) - g(q_0)|}{|g \circ T_{g+\overline{g(q_0)}}(q) + \overline{g(q_0)}|} \leq \frac{|T_{Id+\overline{q_0}}(q) - q_0|}{|T_{Id+\overline{q_0}}(q) + \overline{q_0}|},$$

or

$$(27) \quad \frac{|g(q) - g(q_0)|}{|g(q) + \overline{g(q_0)}|} \leq \frac{|p - q_0|}{|p + \overline{q_0}|} =: r,$$

where

$$(28) \quad p = T_{Id+\overline{q_0}} \circ T_{g^c+g(q_0)}(q).$$

We set

$$h(q) = \frac{g(q) - \operatorname{Im}g(q_0)}{\operatorname{Re}g(q_0)}.$$

Then (27) becomes

$$(29) \quad \frac{|h(q) - 1|}{|h(q) + 1|} \leq r = \frac{|p - q_0|}{|p + \overline{q_0}|}.$$

Consequently,

$$(30) \quad |h(q)| \leq \frac{1+r}{1-r} = \frac{(1+r)^2}{1-r^2} = \frac{(|p + \overline{q_0}| + |p - q_0|)^2}{|p + \overline{q_0}|^2 - |p - q_0|^2}.$$

By the definition of p in (28), we have

$$|p| = |q|, \quad \operatorname{Re}(p) = \operatorname{Re}(q).$$

so that (30) leads to

$$(31) \quad |h(q)| \leq \frac{(|q| + |q_0|)^2}{\operatorname{Re}(q)\operatorname{Re}(q_0)}.$$

This implies that

$$(32) \quad \begin{aligned} |q^{-1}g(q)| &\leq |q^{-1}\operatorname{Im}g(q_0)| + |q^{-1}(g(q) - \operatorname{Im}g(q_0))| \\ &= \frac{|\operatorname{Im}g(q_0)|}{|q|} + |h(q)| \frac{|\operatorname{Re}g(q_0)|}{|q|} \\ &\leq \frac{|\operatorname{Im}g(q_0)|}{|q|} + \frac{(|q| + |q_0|)^2}{\operatorname{Re}(q)\operatorname{Re}(q_0)} \frac{\operatorname{Re}g(q_0)}{|q|}. \end{aligned}$$

Then for all $q \in \mathcal{S}_\gamma$ we have

$$(33) \quad \begin{aligned} |q^{-1}g(q)| &\leq \frac{|\operatorname{Im}g(q_0)|}{|q|} + \frac{(|q| + |q_0|)^2}{\gamma|q|^2} \frac{\operatorname{Re}g(q_0)}{\operatorname{Re}(q_0)} \\ &= \frac{|\operatorname{Im}g(q_0)|}{|q|} + \frac{1}{\gamma} \left(1 + \frac{|q_0|}{|q|}\right)^2 \frac{\operatorname{Re}g(q_0)}{\operatorname{Re}(q_0)}. \end{aligned}$$

By the definition of g in (26), it is evident that

$$\inf \left\{ \frac{\operatorname{Re}g(q)}{\operatorname{Re} q} : q \in \mathbb{H}^+ \right\} = 0.$$

For any $\epsilon > 0$, there thus exists a $q_0 \in \mathbb{H}^+$ such that

$$\frac{\operatorname{Re} g(q_0)}{\operatorname{Re}(q_0)} \leq \gamma \epsilon,$$

and hence by letting $|q| \rightarrow \infty$ in (33) we obtain

$$\limsup_{\substack{|q| \rightarrow \infty \\ q \in \mathcal{S}_\gamma}} |q^{-1}g(q)| \leq \epsilon.$$

Namely,

$$(34) \quad \lim_{\substack{|q| \rightarrow \infty \\ q \in \mathcal{S}_\gamma}} q^{-1}g(q) = 0,$$

since ϵ is an arbitrary positive number. At the same time, for all $q \in \mathcal{S}_\gamma$,

$$\frac{\operatorname{Reg}(q)}{\operatorname{Re}(q)} \leq \frac{|g(q)|}{|q|} \frac{|q|}{\operatorname{Re}(q)} \leq \frac{1}{\gamma} |q^{-1}g(q)|,$$

which together with (34) implies that

$$\lim_{\substack{|q| \rightarrow \infty \\ q \in \mathcal{S}_\gamma}} \frac{\operatorname{Reg}(q)}{\operatorname{Re}(q)} = 0.$$

Now the proof of the assertions (i) and (ii) is complete.

It remains to prove (iii). For any given $\gamma \in (0, 1)$, it is easy to see that the closed ball

$$\overline{\mathbb{B}}(q, \gamma|q|) \subset \mathcal{S}_\gamma,$$

whenever $q \in \mathcal{S}_{2\gamma}$. For any $I \in \mathbb{S}$ and $q \in \mathcal{S}_{2\gamma} \cap \mathbb{C}_I$ we have

$$(35) \quad f'(q) - c = \frac{1}{2\pi I} \int_{\partial \mathbb{B}(q, \gamma|q|) \cap \mathbb{C}_I} \frac{ds}{(s - q)^2} (f(s) - cs).$$

Notice that for any $s \in \partial \mathbb{B}(q, \gamma|q|) \cap \mathbb{C}_I$ we have $|s - q| = \gamma|q|$ so that

$$(1 - \gamma)|q| \leq |s| \leq (1 + \gamma)|q|.$$

Since $|f(s) - cs| = |s| |s^{-1}f(s) - c|$ and

$$\partial \mathbb{B}(q, \gamma|q|) \cap \mathbb{C}_I \subseteq \{s \in \mathcal{S}_\gamma : |s| \geq (1 - \gamma)|q|\}$$

it follows from (35) that

$$(36) \quad |f'(q) - c| \leq \left(1 + \frac{1}{\gamma}\right) \sup_{\substack{|s| \geq (1-\gamma)|q| \\ s \in \mathcal{S}_\gamma}} |s^{-1}f(s) - c|,$$

which approaches to 0 as q tends to ∞ in $\mathcal{S}_{2\gamma}$ due to assertion (ii). Consequently,

$$\lim_{\substack{|q| \rightarrow \infty \\ q \in \mathcal{S}_{2\gamma}}} f'(q) = c.$$

Now the proof is complete. \square

Remark 2. There exists a regular function $f : \mathbb{H}^+ \rightarrow \mathbb{H}^+$ such that

$$c := \inf \left\{ \frac{\operatorname{Re} f(q)}{\operatorname{Re}(q)} : q \in \mathbb{H}^+ \right\} = 0.$$

A simple example is the constant function $f(q) = 1$.

As a simple consequence of the preceding theorem, we have the following results.

Corollary 2. *Let $f : \mathbb{H}^+ \rightarrow \mathbb{H}^+$ be a regular function. Then there exists a positive number β , finite or infinite, such that for any $\gamma > 0$ we have*

$$\lim_{\substack{|q| \rightarrow \infty \\ q \in \mathcal{S}_\gamma}} qf(q) = \beta.$$

Proof. By assumption, f has no zeros and so does f^{-*} (see Proposition 3.9 in [20]). The result immediately follows by applying the preceding theorem to f^{-*} . \square

Another consequence is the following rigidity theorem.

Corollary 3. *Let $f : \mathbb{H}^+ \rightarrow \overline{\mathbb{H}}^+$ be a regular function. If there exist a $I \in \mathbb{S}$ and some $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ such that*

$$\liminf_{r \rightarrow \infty} r |f(re^{I\theta})| = 0,$$

then $f \equiv 0$.

Finally, we give the proof of the Burns-Krantz theorem.

Proof of Theorem 3. Let φ be the Cayley transformation from \mathbb{H}^+ to \mathbb{B} , i.e.

$$\varphi(q) = (1 + q)^{-1}(1 - q), \quad \forall q \in \mathbb{H}^+.$$

The result immediately follows by applying the preceding corollary to the regular function $g = f \circ \varphi$. \square

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